1.1b) Pseudoinverse

Compute the Pseudoinverse of

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

begin by calculating the singular value decomposition

$$A \cdot A^T = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} \begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 25 & 50 \\ 50 & 100 \end{pmatrix}$$

$$p(\lambda) = \begin{Vmatrix} 25-\lambda & 50 \\ 50 & 100-\lambda \end{Vmatrix}$$

$$= (25-\lambda)(100-\lambda) - 2500$$

$$= \lambda^2 - 125\lambda = \lambda(\lambda - 125)$$

$$\Rightarrow \lambda_1 = 125 \quad , \quad \lambda_2 = 0$$

Calculating the eigenpairs

$$\lambda_1 : \quad \begin{aligned} -100x_1 + 50x_2 &= 0 \\ 50x_1 - 25x_2 &= 0 \end{aligned} \quad \Rightarrow \text{normalized eigenvector} \quad \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 : \quad \begin{aligned} 25x_1 + 50x_2 &= 0 \\ 50x_1 + 100x_2 &= 0 \end{aligned} \quad \Rightarrow \text{normalized eigenvector} \quad \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

We hence have

$$\Sigma = \begin{pmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 5\sqrt{5} & 0 \\ 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \Rightarrow \cancel{V^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}}$$

1.1 b Pseudoinverse continued

$$A^T A = \begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 80 & 60 \\ 60 & 45 \end{pmatrix}$$

$$p_2(\lambda) = \begin{Vmatrix} 80 - \lambda & 60 \\ 60 & 45 - \lambda \end{Vmatrix}$$

$$= (80 - \lambda)(45 - \lambda) - 3600$$

$$= \lambda^2 - 125 \lambda = \lambda(\lambda - 125)$$

Calculating the eigenpairs

$$\lambda_1 : \quad \begin{matrix} -45 x_1 + 60 x_2 = 0 \\ 60 x_1 - 80 x_2 = 0 \end{matrix} \quad \Rightarrow \text{ normalized eigenvectors } \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\lambda_2 : \quad \begin{matrix} 80 x_1 + 60 x_2 = 0 \\ 60 x_1 + 80 x_2 = 0 \end{matrix} \quad \Rightarrow \text{ normalized eigenvectors } \frac{1}{5}\begin{pmatrix} -3 \\ +4 \end{pmatrix}$$

We hence have

$$V = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & +4 \end{pmatrix}$$

The singular value decomposition is then

$$\cancel{\begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 125 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}}$$

Pseudoinverse continued II

The singular value decomposition is hence

$$\begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} 5\sqrt{5} & 0 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \frac{4}{5} & +\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}}_{V^T}$$

~~according~~ our pseudoinverse is then given by

$$A^+ = V \Sigma^+ U^*$$

$$\Rightarrow A^+ = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{5\sqrt{5}} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{125} & \frac{8}{125} \\ \frac{3}{125} & \frac{6}{125} \end{pmatrix}$$

$$= \frac{1}{125}\begin{pmatrix} 4 & 8 \\ 3 & 6 \end{pmatrix}$$

1.2 a) Singular Value Decomposition

Compute the singular value decomposition of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Compute A^TA

$$A^TA = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

$$p(\lambda) = \det \begin{pmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{pmatrix}$$

$$= -\lambda^3 + 22\lambda^2 - 120\lambda$$

$$= \lambda(-\lambda^2 + 22\lambda - 120)$$

$$= -\lambda(\lambda^2 - 22\lambda + 120)$$

$$= -\lambda(\lambda - 10)(\lambda - 12)$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 10, \quad \lambda_3 = 12$$

~~Basis for the nullspace given by~~

Finding the eigenvectors

$$\lambda_1 : \left[\begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{aligned} 10x_1 + \quad\quad + 2x_3 &= 0 \\ 10x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 + 2x_3 &= 0 \end{aligned}$$

Using Gaussian elimination

$$x_1 = -1 \quad , \quad x_3 = 5 \quad \Rightarrow x_2 = -2$$

Normalizing the eigenvector

$$\Rightarrow \frac{1}{\sqrt{30}} \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

$$\lambda_2: \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$$

$$2x_3 = 0$$
$$4x_3 = 0$$
$$2x_1 + 4x_2 - 8x_3 = 0$$

Using Gaussian elimination

$$x_3 = 0, \quad x_1 = -2, \quad x_2 = \cancel{0}\,1$$

Normalizing the eigenvector

$$\Rightarrow \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3: \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \begin{aligned} -2x_1 + \quad + 2x_3 &= 0 \\ -2x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - 10x_3 &= 0 \end{aligned}$$

Using Gaussian elimination

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 1$$

Normalizing the eigenvector

$$\Rightarrow \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Our eigenpairs are hence

$$\left[12,\ \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}\right],\ \left[10,\ \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix}\right],\ \left[0,\ \frac{1}{\sqrt{30}}\begin{pmatrix}-1\\-2\\5\end{pmatrix}\right]$$

$$\Rightarrow \quad \Sigma = \begin{pmatrix}\sqrt{12} & 0 & 0\\ 0 & \sqrt{10} & 0\end{pmatrix}$$

$$\Rightarrow \quad V = \begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}}\\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}}\\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}}\end{pmatrix}$$

$$\Rightarrow \quad V^T = \begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0\\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}}\end{pmatrix}$$

As the singular value decomposition theorem states

$$A_{n\times p} = U_{n\times n}\,\Sigma_{n\times p}\,V^T_{p\times p}$$

$$\Rightarrow \quad n = 2$$

$$\cancel{AA}\ AA^T = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}$$

$$\Rightarrow p_2(\lambda) \begin{Vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{Vmatrix}$$

$$= (11-\lambda)^2 - 1$$

$$= 120 - 22\lambda + \lambda^2$$

$$= (\lambda - 12)(\lambda - 10)$$

Calculating the eigenpairs

$\lambda_1 = 12$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0} \Rightarrow$$ the normalized eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 10$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0} \Rightarrow$$ the normalized eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix} +1 \\ -1 \end{pmatrix}$

the eigenpairs are hence

$$\left[12, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right], \left[10, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right]$$

$$\Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

the singular value decomposition is hence

$$\begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{pmatrix}$$

2.5 Mean, Mode, Variance for the Beta Distribution

Suppose $\theta \sim \mathrm{Beta}(a,b)$. Derive the mean, mode and variance.

Mean:

$$\mu = \mathbb{E}[\theta] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \underbrace{\int_0^1 \theta^{a}(1-\theta)^{b-1}\,d\theta}_{= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}}$$

$$= \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

using the identity of the Gamma function

$$\Gamma(a+1) = a\,\Gamma(a)$$

to simplify the expression above

$$= \frac{a\,\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}}{(a+b)\,\cancel{\Gamma(a+b)}} \cdot \frac{\cancel{\Gamma(a+b)}}{\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}}$$

$$= \frac{a}{a+b}$$

2.5 continued

mode:

$$p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,\theta^{a-1}(1-\theta)^{b-1}$$

$$\frac{d}{d\theta}p(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\Big((a-1)\theta^{a-2}(1-\theta)^{b-1} - (b-1)\theta^{a-1}(1-\theta)^{b-2}\Big)$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\,\theta^{a-2}(1-\theta)^{b-2}\Big((a-1)(1-\theta)-(b-1)\theta\Big)$$

find the maximum, as taught in differential calculus

$$p'(\theta) = 0$$

$$\Rightarrow \quad (a-1)(1-\theta) - (b-1)\theta = 0$$

$$a-1 - \theta(a-1+b-1) = 0$$

$$\Rightarrow \quad \theta = \frac{a-1}{a+b-2}$$

variance:

$$\mathrm{Var}(\theta) = \mathbb{E}[\theta^2] - \mathbb{E}[\theta]^2$$

$$\Rightarrow \mu^2 + \sigma^2 = \mathbb{E}[\theta^2]$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \underbrace{\int_0^1 \theta^{a+1}(1-\theta)^{b-1}\,d\theta}_{= \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)}}$$

2.5 continued II

$$= \frac{(a+\overset{1}{\cancel{2}})(a+\cancel{1})\ \Gamma(a)\Gamma(b)}{(a+b+\cancel{2})(a+b+1)\ \Gamma(a+b)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$\mu^2 + \sigma^2 = \frac{a(a+1)}{(a+b)(a+b+1)}$$

using μ from part 1

$$\sigma^2 = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2}$$

$$= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)}$$

$$= \frac{a^3 + a^2b + a^2 + ab - a^3 - a^2b - a^2}{(a+b)^2(a+b+1)}$$

$$= \underline{\underline{\frac{ab}{(a+b)^2(a+b+1)}}}$$